DIFFERENTIAL GERSTENHABER ALGEBRAS AND GENERALIZED DEFORMATIONS OF SOLVMANIFOLDS

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ABSTRACT. We consider differential Gerstenhaber algebras (DGA) of solvmanfiolds with complex stuructures and symplectic structures respectively. We give explicit finite dimensional subDGAs of such DGAs such that the inclusions induce cohomology isomorphisms. By this result, we study the generalized Kuranishi spaces of solvmanifolds with left-invariant complex structures. In particular we extend Rollenske's results of stability of symmetry property of nilmanifolds and of estimations of smoothness of Kuranishi spaces of complex parallelizable nilmanifolds, to spacial classes of solvmanifolds. By the result in DGAs of solvmanifolds, we also give explicit finite dimensional cohcain complexes which computes the holomorphic poisson cohomology of nilmanifolds and solvmanifolds. We also give classes of Pseudo-Kähler solvmanifolds whose mirror images in the sense of Merkulov are themselves.

1. INTRODUCTION

Let (M,J) be a complex manifold. We consider the space $A^{0,*}(M,\bigwedge T_{1,0}M)$ of $(0,*)$-differential forms with values in the holomorphic tangent poly-vector bundle with the Schouten bracket $[\bullet]$ and the Dolbeault operator $\bar\partial$. We denote $dG^{r}(M,J)=\bigoplus_{p+q=r}A^{0,q}(M,\bigwedge^{p}T_{1,0}M)$. Then $(dG^{*}(M,J),\wedge,[\bullet],\bar\partial)$ is a differential Gerstenhaber algebra (shortly DGA).

Let (N,ω) be a symplectic manifold. We consider the 2-vector field ω^{-1} as a poisson structure. Then the space $dG^{*}(N,\omega^{-1})=C^{\infty}(N,\bigwedge TN)$ of the poly-vector fields with the Schouten bracket $[\bullet]$ and the differential $[\omega^{-1}\bullet\ ]: dG^{*}(N,\omega^{-1})\to dG^{*+1}(N,\omega^{-1})$ is a DGA. Regarding ω as the isomorphism $\omega: TN\to T^{*}N$, the space $dG^{*}(N,\omega)=A^{*}(N)$ of the differential forms with the bracket $[\bullet]_{\omega}$ induced by the Schouten bracket is a DGA.

Let G be a simply connected solvable Lie group with a lattice (i.e. cocompact discrete subgroup) Γ. In this paper, we consider the DGA $dG^{*}(G/\Gamma,J)$ (resp. $dG^{*}(G/\Gamma,\omega)$) of a solvmanifold with a complex (resp. symplectic) structure J (resp. ω). In the previous papers [11], [12] [13], the author study the de Rham cohomology of general solvmanifolds and the Dolbeault cohomology of some classes of solvmanifolds with left-invariant complex structures. By using these results, in this paper we construct explicit finite dimensional sub-DGAs of $dG^{*}(G/\Gamma,J)$ and $dG^{*}(G/\Gamma,\omega)$ such that the inclusions induce cohomology isomorphisms. In this paper we apply such sub-DGAs to the following two theory.

1. Generalized deformation. Consider the generalized complex structure $\mathcal{J}$ induced by the complex structure J. By the generalized Kuranishi space constructed by $dG^{*}(M,J)$, we study small deformations of the generalized complex structure $\mathcal{J}$ ([8]). In this paper, by the above results, we see that small deformations of generalized complex structures which are given by left-invariant complex structures on solvmanifolds are controlled by finite dimensional DGAs. In particular, under some conditions, we prove that certain symmetry properties (not

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only left-G-invariance) are preserved under small deformation as similar to Rollenske's results for deformations of complex structures on nilmanifolds in [22] and we can estimate the smoothness of generalized Kuranishi space as similar to Rollenske's results for complex parallelizable nilmanifolds [23].

Moreover, we give examples of non-Kähler complex solvmanifolds with trivial canonical bundles whose generalized Kuranishi spaces are smooth.

2. Holomorphic poisson cohomology Let (M, J) be a compact complex manifold. A bi-vector field $\mu \in C^{\infty}(\bigwedge^2 T_{1,0}M)$ is called holomorphic poisson if $\bar{\partial}\mu = 0$ and $[\mu \bullet \mu] = 0$. Let μ be a holomorphic poisson bi-vector field μ on M. Then we can define the cohomology determined by μ which is isomorphic to the Lie algebroid cohomology of generalized complex structure given by μ (see [15]). Unlike real poisson cohomology, this cohomology is always finite dimensional (see [15]). In this paper, by the results in DGAs of solvmanifolds, we show that the holomorphic poisson cohomologies of nilmanifolds and solvmanifolds are computed by some finite dimensional cochain complexes.

3. Weak mirror symmetry. A pair of a complex manifold (M, J) and a symplectic manifold (N, ω) is called a weak mirror pair if two DGAs $dG^r(M, J)$ and $dG^*(N, \omega) \otimes \mathbb{C}$ are quasi-isomorphic ([18],[17]). We are interested in a pseudo-Kähler manifold (M, ω, J) such that "self mirror" occurs i.e. (M, ω) and (M, J) is a weak mirror pair. Pseudo-Kähler nilmanifolds with self mirror have been found in [20], [2] and [3]. In this paper, we give classes of Pseudo-Kähler solvmanifolds with self mirror.

2. Notation and Conventions

- "DGA" means "differential Gerstenhaber algebra".
- "DGrA" means "differential graded algebra". (We must distinguish DGA and DGrA.)
- "DBiA" means "differential bi-graded algebra".
- We write $[n] = \{1, 2, \dots, n\} = \{a \in \mathbb{Z}_{>0} | a \leq n\}$.
- For a multi-index $I = \{i_1, \dots, i_p\}$, products through I are written shortly, for examples, $\alpha_I = \alpha_{i_1} \cdot \alpha_{i_2} \dots \alpha_{i_p}$, $x_I = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$ etc.

3. DGAs of Lie algebras

Let G be a simply connected Lie algebra with a left-invariant complex structure J. and $\mathfrak{g}$ the Lie algebra of G. Consider the decomposition $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1}$ where $\mathfrak{g}_{1,0}$ (resp. $\mathfrak{g}_{0,1}$) is the $\sqrt{-1}$-eigenspace (resp. $\sqrt{-1}$-eigenspace) of J. Then we can define the bracket $[\bullet]$ on $\mathfrak{g} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}^*_{0,1}$ as

$$[X + \bar{x} \bullet Y + \bar{y}] = [X, Y] + i_X d\bar{y} - i_Y d\bar{x}$$

for $X, Y \in \mathfrak{g}_{1,0}$, $\bar{x}, \bar{y} \in \mathfrak{g}^*_{0,1}$ where d is the exterior differential which is dual to the Lie bracket. Extending this bracket on $\bigwedge(\mathfrak{g}_{1,0} \oplus \mathfrak{g}^*_{1,0})$ and considering the Dolbeault operator $\bar{\partial}$, $(\bigwedge(\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,0*}), \wedge, [\bullet], \bar{\partial})$ is a DGA. We denote this DGA by $dG^*(\mathfrak{g}, J)$.

Remark 1. Suppose (G, J) is a complex Lie group. Then since $\mathfrak{g}_{1,0}$ and $\mathfrak{g}^*_{0,1}$ consist of holomorphic vector fields and anti-holomorphic forms respectively, we have $i_X d\bar{y} = 0$ and $\bar{\partial}X = 0$ for $X \in \mathfrak{g}_{1,0}$ and $\bar{y} \in \mathfrak{g}^*_{0,1}$. Hence we regarding $\bigwedge \mathfrak{g}_{1,0}$ as a DGA with trivial differential and $\bigwedge \mathfrak{g}^*_{0,1}$ as a DGA with trivial bracket, we have

$$dG^*(\mathfrak{g}, J) = \bigwedge \mathfrak{g}_{1,0} \otimes \bigwedge \mathfrak{g}^*_{0,1}$$

as a tensor product of DGAs. In particular we have

$$[\bigwedge^p \mathfrak{g}_{1,0} \otimes \bigwedge^q \mathfrak{g}^*_{0,1} \bullet \bigwedge^0 \mathfrak{g}_{1,0} \otimes \bigwedge^r \mathfrak{g}^*_{0,1}] = 0.$$

We suppose G has a lattice (i.e. cocompact discrete subgroup) Γ. Then we have the inclusion $dG^*(\mathfrak{g},J)\subset dG^*(G/\Gamma,J)$ of DGA.

We use the following theorem:

Theorem 3.1. ([1, Theorem 3.1]) *Let G be a simply connected $2n$-dimensional nilpotent Lie group with a left invariant structure J. Then we have $d(\bigwedge^n \mathfrak{g}^*_{1,0})=0$. In particular for a lattice Γ, the canonical bundle $\bigwedge^n T^*_{1,0}G/\Gamma$ of the nilmanifold G/Γ is trivial.*

By this theorem we have

Corollary 3.2. *Let G be a simply connected $2n$-dimensional nilpotent Lie group with a left invariant structure J. We suppose that the inclusion $\bigwedge^* \mathfrak{g}_{1,0}\otimes \bigwedge^* \mathfrak{g}^*_{0,1}\subset A^{*,*}(G/\Gamma)$ induces an isomorphism*

$$H^{*,*}_{\bar\partial}(\mathfrak{g})\cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

Then the inclusion $dG^*(\mathfrak{g},J)\subset dG^*(G/\Gamma,J)$ induces a cohomology isomorphism.

Proof. Fix $p\in\mathbb{N}$. We consider the natural $\mathbb{C}$-linear isomorphism $\bigwedge^p \mathfrak{g}_{1,0}\cong \bigwedge^{n-p}\mathfrak{g}^*_{1,0}$ given by the pairing

$$\bigwedge^{p}\mathfrak{g}^*_{1,0}\times \bigwedge^{n-p}\mathfrak{g}^*_{1,0}\to \bigwedge^{n}\mathfrak{g}^*_{1,0}.$$

By theorem 3.1, this isomorphism induces an isomorphism $\bigwedge^p T_{1,0}G/\Gamma\cong \bigwedge^{n-p}T^*_{1,0}G/\Gamma$ of holomorphic vector bundles, and hence induces an isomorphism

$$(A^{0,*}(G/\Gamma,\bigwedge^{p}T_{1,0}G/\Gamma),\bar\partial)\cong (A^{n-p,*}(G/\Gamma),\bar\partial).$$

Now we have the commutative diagram

$$\begin{tikzcd}
\bigwedge^{n-p}\mathfrak{g}^*_{1,0}\otimes\bigwedge^*\mathfrak{g}_{0,1} \arrow[r] \arrow[d, "\cong"] & A^{n-p,*}(G/\Gamma) \arrow[d, "\cong"] \\
\bigwedge^{p}\mathfrak{g}_{1,0}\otimes\bigwedge^*\mathfrak{g}^*_{0,1} \arrow[r] & A^{0,*}(G/\Gamma,\bigwedge^p T_{1,0}G/\Gamma).
\end{tikzcd}$$

Since the inclusion $\bigwedge^{n-p}\mathfrak{g}^*_{1,0}\otimes\bigwedge^*\mathfrak{g}_{0,1}\subset A^{n-p,*}(G/\Gamma)$ induces a cohomology isomorphism, the inclusion $\bigwedge^p\mathfrak{g}_{1,0}\otimes\bigwedge^*\mathfrak{g}^*_{0,1}\subset A^{0,*}(G/\Gamma,\bigwedge^p T_{1,0}G/\Gamma)$ does so. Hence the corollary follows. $\square$

Theorem 3.3. *Let G be a simply connected nilpotent Lie group with a lattice Γ and a left-invariant complex structure J. Then the inclusion $\bigwedge^*\mathfrak{g}_{1,0}\otimes\bigwedge^*\mathfrak{g}^*_{0,1}\subset A^{*,*}(G/\Gamma)$ induces an isomorphism*

$$H^{*,*}_{\bar\partial}(\mathfrak{g})\cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

if (G,J,Γ) meet one of the following conditions:

(N) The complex manifold $(G/\Gamma,J)$ has the structure of an iterated principal holomorphic torus bundle ([5]).

(Q) J is a small deformation of a rational complex structure i.e. for the rational structure $\mathfrak{g}_{\mathbb{Q}}\subset\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ induced by a lattice Γ (see [21, Section 2]) we have $J(\mathfrak{g}_{\mathbb{Q}})\subset\mathfrak{g}_{\mathbb{Q}}$ ([4]).

(C) (G,J) is a complex Lie group ([24]).

4. Generalized deformations

4.1. **DGA and generalized Kuranishi space.** Let (M,J) be a compact n-dimensional complex manifold. We define the generalized complex structure $\mathcal{J}\in \mathrm{End}(TM\oplus T^{*}M)$ by $\mathcal{J}=\begin{pmatrix} -J & 0\\ 0 & J^{*}\end{pmatrix}$ with the maximally isotropic subspace $L_{\mathcal{J}}=T_{0,1}M\oplus T^{*}_{1,0}M$ in $(TM\oplus T^{*}M)\otimes\mathbb{C}$.

For $\epsilon\in C^{\infty}(\bigwedge^{2}L^{*})=C^{\infty}(\bigwedge^{2}\bar{L})=dG^{2}(M,J)$, we have the new generalized complex structure given by the maximally isotropic subspace

$$L_{\epsilon}=(1+\epsilon)L_{\mathcal{J}}=\{E+i_{E}\epsilon | E\in L_{\mathcal{J}}\}$$

if ϵ satisfies the generalized Maurer-Cartan equation

$$\bar{\partial}\epsilon+\frac{1}{2}[\epsilon\bullet\epsilon]=0.$$

For a Hermitian metric on M, we consider the $\mathbb{C}$-anti-linear Hodge star operator

$$\bar{*}: A^{0,q}(M,\bigwedge^{p}T_{1,0}M)\to A^{n,n-q}(M,\bigwedge^{p}T^{*}_{1,0}M),$$

the adjoint differential operator $\bar{\partial}^{*}=-\bar{*}\bar{\partial}\bar{*}$, the Laplacian operator $\Box_{g}=\bar{\partial}^{*}\bar{\partial}+\bar{\partial}\bar{\partial}^{*}$, the space $\mathcal{H}^{*}_{g}(M,J)=\mathrm{Ker}\,\Box_{g}|_{dG^{*}(M,J)}$ of harmonic forms, the orthogonal projection $H: dG^{*}(M,J)\to \mathcal{H}^{*}_{g}(M,J)$ and the Green operator $G: dG^{*}(M,J)\to dG^{*}(M,J)$. Take a basis $\eta_{1},\dots\eta_{k}$ of $\mathcal{H}^{*}_{g}(M,J)$. For parameters $t=(t_{1},\dots,t_{k})$, we consider the formal power series $\phi(t)=\phi(t_{1},\dots,t_{k})$ with values in $dG^{2}(G/\Gamma,J)$ given inductively by $\phi_{1}(t)=\sum_{i=1}^{k}t_{i}\eta_{i}$ and

$$\phi_{r}(t)=\frac{1}{2}\sum_{s=1}^{r-1}\bar{\partial}^{*}G[\phi_{s}(t)\bullet\phi_{r-s}(t)].$$

For sufficiently small $\delta>0$, for $|t|<\delta$, $\phi(t)$ converges. We denote

$$Kur^{gen}(M,J)=\{t=(t_{1},\dots,t_{k})|\ |t|<\delta,\ H([\phi(t)\bullet\phi(t)])=0\}$$

Then we have:

Theorem 4.1. ([8]) *For $t\in Kur^{gen}(M,J)$, $\phi(t)$ satisfies the generalized Maurer-Cartan equation. Any small deformation of the generalized complex structure of $\mathcal{J}$ is equivalent to a generalized complex structure given by the maximally isotropic subspace*

$$L_{\phi(t)}=(1+\phi(t))L_{\mathcal{J}}$$

for some $t\in Kur^{gen}(M,J)$.

Suppose the canonical bundle $\bigwedge^{n}T^{*}_{1,0}M$ is trivial. Since we have an isomorphism $\bigwedge^{p}T^{*}_{1,0}M\cong\bigwedge^{n-p}T_{1,0}M$ by the pairing $\wedge:\bigwedge^{p}T^{*}_{1,0}M\times\bigwedge^{n-p}T^{*}_{1,0}M\to\bigwedge^{n}T^{*}_{1,0}M$, we can identifies $(A^{n,*}(M,\bigwedge^{p}T^{*}_{1,0}M),\bar{\partial})$ with $(A^{0,*}(M,\bigwedge^{n-p}T_{1,0}M),\bar{\partial})$. Hence we can regard

$$\bar{*}_{g}: A^{0,q}(M,\bigwedge^{p}T_{1,0}M)\to A^{0,n-q}(M,\bigwedge^{n-p}T_{1,0}M)$$

and so

$$\bar{*}_{g}: dG^{r}(M,J)\to dG^{2n-r}(M,J)$$

Lemma 4.2. *Suppose the canonical bundle $\bigwedge^{n}T^{*}_{1,0}M$ is trivial. Let C^{*} be a finite-dimensional subDGA of $dG(M,J)$ such that $\bar{*}(C^{*})\subset C^{*}$. Suppose the inclusion $C^{*}\subset dG^{*}(M,J)$ induces a cohomology isomorphism. Then for $t\in Kur^{gen}(M,J)$, we have*

$$\phi(t)\in C^{2}.$$

Proof. By $\bar{*}(C^*) \subset C^*$, we can regard $\partial^*_{|C^*}$ and $\Box_{g|C^*}$ as operators on C^*. Since C^* is finite dimensional, we can easily show the decomposition

$$C^* = \operatorname{Ker} \Box_{g|C^*} \oplus \operatorname{Im} \partial_{|C^*} \oplus \operatorname{Im} \partial^*_{|C^*}.$$

Hence the induced map $H^*(C^*) \to H^*(dG^*(M,J))$ is represented by the inclusion $\operatorname{Ker} \Box_{|C^*} \subset \mathcal{H}^*(M,J)$. Since the induced map $H^*(C^*) \to H^*(dG^*(M,J))$ is an isomorphism, we have $\operatorname{Ker} \Box_{|C^*} = \mathcal{H}^*(M,J)$. For $t_i\eta_i \in \mathcal{H}^2(M,J) = \operatorname{Ker} \Box_{|C^*}$, since C^* is a finite-dimensional sub-DGA of $dG^*(M,J)$, for a basis $\zeta_1, \dots, \zeta_l$ of C^2, we have $\phi(t) = \sum \psi_i(t)\zeta_i$ which converges for $|t| < \delta$. hence the lemma follows. $\Box$

By this lemma we can prove generalized version of Rollenske's result in [22].

Theorem 4.3. *Let G be a simply connected $2n$-dimensional nilpotent Lie group with a left invariant structure J. We suppose that the inclusion $\bigwedge^* \mathfrak{g}^*_{1,0} \otimes \bigwedge^* \mathfrak{g}^*_{0,1} \subset A^{*,*}(G/\Gamma)$ induces an isomorphism*

$$H^{*,*}_{\bar\partial}(\mathfrak{g}) \cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

Let $\mathcal{J}$ be the generalized complex structure given by J. Then any small deformation of $\mathcal{J}$ is equivalent to a generalized complex structure which is induced by a left-invariant generalized complex structure on G.

Proof. Take a basis $X_1, \dots, X_n$ of $\mathfrak{g}_{1,0}$. We consider the left-invariant Hermitian metric $g = \sum x_i \bar{x}_i$ where $x_1, \dots, x_n$ is the basis of $\mathfrak{g}^*_{1,0}$. Then we have $\bar{*}_g(dG^*(\mathfrak{g},J)) \subset dG^*(\mathfrak{g},J)$. For the basis $X_1, \dots, X_n, \bar{X}_1, \dots, \bar{X}_n, x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$ of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathbb{C}$ we write

$$\mathcal{J} = \sqrt{-1} \sum (X_i \otimes x_i + \bar{x}_i \otimes \bar{X}_i) - \sqrt{-1} \sum (\bar{X}_i \otimes \bar{x}_i + x_i \otimes X_i)$$

By Theorem 4.1 and Lemma 4.2, any small deformation of $\mathcal{J}$ is equivalent to

$$\mathcal{J} = \sqrt{-1} \sum (X_i^\epsilon \otimes x_i^\epsilon + (\bar{x}_i)^\epsilon \otimes (\bar{X}_i)^\epsilon) - \sqrt{-1} \sum (\bar{X}_i^{\bar\epsilon} \otimes \bar{x}_i^{\bar\epsilon} + x_i^{\bar\epsilon} \otimes X_i^{\bar\epsilon})$$

where we write $E^\epsilon = E + i_E\epsilon$ for $E \in L_{\mathcal{J}}$. Hence the theorem follows. $\Box$

4.2. **Radius of convergence and smoothness.** We can prove generalized version of Rollenske's result in [23].

Theorem 4.4. *Let G be a n-dimensional simply connected complex nilpotent Lie group with a lattice Γ. Suppose G is ν-step i.e. $C_{\nu-1}\mathfrak{g} = 0$ and $C_\nu \mathfrak{g} = 0$ where we denote by $\mathfrak{g} = C_0\mathfrak{g} \supset C_1\mathfrak{g} \supset C_2\mathfrak{g} \cdots$ the lower central series. Then $\phi(t)$ is a finite sum*

$$\phi(t) = \sum_{i=1}^{\nu} \phi_i(t)$$

and we have

$$H[\phi(t), \phi(t)] = \sum_{i+j \le \nu} H[\phi_i(t), \phi_j(t)].$$

In particular $\delta = \infty$ and $Kur^{gen}(G/\Gamma, J)$ is cut out by polynomial equations of degree at most ν.

Proof. By remark 1, we have

$$
\begin{aligned}
&[dG^2(\mathfrak{g},J)\bullet dG^2(\mathfrak{g},J)]\\
&\subset [\bigwedge^{1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}\bullet\bigwedge^{1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}]\bigoplus[\bigwedge^{2}\mathfrak{g}_{1,0}\otimes\bigwedge^{0}\mathfrak{g}^{*}_{0,1}\bullet\bigwedge^{1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}]\\
&\qquad\bigoplus[\bigwedge^{2}\mathfrak{g}_{1,0}\otimes\bigwedge^{0}\mathfrak{g}^{*}_{0,1}\bullet\bigwedge^{2}\mathfrak{g}_{1,0}\otimes\bigwedge^{0}\mathfrak{g}^{*}_{0,1}].
\end{aligned}
$$

We have $\bar\partial^{*}(\bigwedge^{p}\mathfrak{g}_{1,0})=0$ for any $p\in\mathbb{N}$. Moreover since we have $\bar\partial_{|\bigwedge^{n-1}\mathfrak{g}^{*}_{0,1}}=0$ by the unimodularity of G, we have $\bar\partial^{*}(\bigwedge^{p}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1})=0$ for any $p\in\mathbb{N}$. Hence we have

$$\bar\partial^{*}G[dG^2(\mathfrak{g},J)\bullet dG^2(\mathfrak{g},J)]\subset[\bigwedge^{1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}\bullet\bigwedge^{1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}].$$

By this, for $i\geq 2$ inductively we have

$$\phi_i(t)\in C_{i-1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}$$

for $i\geq 2$. Hence we have $\phi_{\nu+1}(t)=0$ and $[\phi_i(t),\phi_j(t)]\in C_{i+j-1}\mathfrak{g}_{1,0}\otimes\bigwedge^{1}\mathfrak{g}^{*}_{0,1}$ for $i,j\geq 2$ and $[\phi_\nu(t),\phi_1(t)]=0$. These implies the theorem. □

We consider the special condition which implies $\delta=\infty$ and $Kur^{gen}(M,J)$ is smooth. In Section 9, we give a large class of complex solvmanifolds satisfying such condition

Proposition 4.5. *Let (M,J) be a compact complex manifold. Suppose there exists a Hermitian metric g on M such that the space $\mathcal{H}_g(M,J)$ is closed under the Schouten bracket. Take a basis $\eta_1,\dots\eta_k$ of $\mathcal{H}^{*}_g(M,J)$. Then we have*

$$Kur^{gen}(M,J)=\{t=(t_1,\dots,t_k)|[\sum_{i=1}^{k}t_i\eta_i\bullet\sum_{i=1}^{k}t_i\eta_i]=0\}$$

In particular $\delta=\infty$ and $Kur^{gen}(G/\Gamma,J)$ is cut out by polynomial equations of degree at most 2.

Moreover suppose the Schouten bracket on $\mathcal{H}^{*}_g(M,J)$ is trivial. Then $Kur^{gen}(G/\Gamma,J)$ is smooth.

Proof. For any parameter $t=(t_1,\dots,t_k)$ we have $[\sum_{i=1}^{k}t_i\eta_i\bullet\sum_{i=1}^{k}t_i\eta_i]\in\mathcal{H}^{*}_g(M,J)$. Hence we have $G[\sum_{i=1}^{k}t_i\eta_i\bullet\sum_{i=1}^{k}t_i\eta_i]=0$ and so we have $\phi(t)=\sum_{i=1}^{k}t_i\eta_i$. This implies the first assertion of the proposition. Obviously, the second assertion follows from the first assertion. □

Remark 2. For a compact complex manifold, we consider the differential graded Lie algebra

$$A^{0,*}(M,T_{1,0}M)$$

of differential forms with values in the holomorphic tangent bundle and the space

$$\mathcal{H}^{*}_g(M,J)\cap A^{0,*}(M,T_{1,0}M)$$

of harmonic forms which belong to such sapce. We take a basis a basis $\eta_1,\dots,\eta_j,\dots\eta_k$ of $\mathcal{H}^{2}_g(M,J)$ such that $\eta_1,\dots,\eta_j$ is a basis of $\mathcal{H}^{2}_g(M,J)\cap A^{0,1}(M,T_{1,0}M)$. Then the subspace

$$Kur(M,J)=\{t=(t_1,\dots,t_j,0,\dots,0)|\ |t|<\delta,\ H([\phi(t)\bullet\phi(t)])=0\}$$

of $Kur(M,J)^{gen}$ is the usual Kuranishi space (see [14] and [8]). Hence study of generalized deformation covers study of usual deformation of complex structures.

5. Cohomology of Holomorphic poisson manifolds

Let (M,J) be a compact complex manifold. A bi-vector field $\mu\in C^{\infty}(\bigwedge^{2}T_{1,0}M)$ is called holomorphic poisson if $\bar\partial\mu=0$ and $[\mu\bullet\mu]=0$. Let μ be a holomorphic poisson bi-vector field μ on M. Then $T^{*}M$ is naturally a holomorphic Lie algebroid. Since $\mu\in dG^{*}(M,J)$ satisfies the generalized Maurer-Cartan equation, we have the deformed generalized structure given by the maximally isotropic subspace L_{μ}. Considering the DBiA $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial)$, by the differential operator $[\mu\bullet]: A^{0,*}(M,\bigwedge^{*}T_{1,0}M)\to A^{0,*}(M,\bigwedge^{*+1}T_{1,0}M)$, we have the double complex $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial,[\mu\bullet])$. Then we consider the following three cohomology:

- The Lie algebroid cohomology of the holomorphic Lie algebroid $T^{*}M$.
- The Lie algebroid cohomology of the algebroid L_{μ} (Lie algeboid cohomology of a generalized complex manifold see [8]).
- The total cohomology of the double complex $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial,[\mu\bullet])$.

It is known that, these cohomology are all isomorphic (see [15]). We denote by $H^{*}(M,\mu)$ one of these cohomology. We call it the holomorphic poisson cohomology.

Lemma 5.1. *Let $C^{*,*}$ be a sub-DBiA of the DBiA $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial)$. We suppose that the inclusion $C^{*,*}\subset A^{0,*}(M,\bigwedge^{*}T_{1,0}M)$ induces a $\bar\partial$-cohomology isomorphism and for $C^{r}=\bigoplus_{p+q=r}C^{p,q}$, C^{*} is a sub-DGA of $dG^{*}(M,J)$. Let $\mu\in C^{2,0}$ be a holomorphic poisson bi-vector field. Then the inclusion $C^{*,*}\subset A^{0,*}(M,\bigwedge^{*}T_{1,0}M)$ induces an isomorphism between the total cohomology of $(C^{*,*},\bar\partial,[\mu\bullet])$ and the total cohomology of $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial,[\mu\bullet])$. Hence the total cohomology of $(C^{*,*},\bar\partial,[\mu\bullet])$ is isomorphic to $H^{*}(M,\mu)$.*

Proof. For the double complexes $(C^{*,*},\bar\partial,[\mu\bullet])$ and $(A^{0,*}(M,\bigwedge^{*}T_{1,0}M),\bar\partial,[\mu\bullet])$, we have the spectral sequences $E^{*,*}_{*}(C^{*,*})$ and $E^{*,*}_{*}(A^{0,*}(M,\bigwedge^{*}T_{1,0}M))$ such that $E^{*,*}_{1}(C^{*,*})\cong H^{*,*}_{\bar\partial}(C^{*,*})$ and $E^{*,*}_{1}(A^{0,*}(M,\bigwedge^{*}T_{1,0}M))\cong H^{*,*}_{\bar\partial}(M,A^{0,*}(M,\bigwedge^{*}T_{1,0}M))$. Since the inclusion $C^{*,*}\subset A^{0,*}(M,\bigwedge^{*}T_{1,0}M)$ induces a $\bar\partial$-cohomology isomorphism, the inclusion $C^{*,*}\subset A^{0,*}(M,\bigwedge^{*}T_{1,0}M)$ induces an isomorphism $E^{*,*}_{1}(C^{*,*})\cong E^{*,*}_{1}(A^{0,*}(M,\bigwedge^{*}T_{1,0}M))$. Hence by [16, Theorem 3.5], the lemma follows. □

By this lemma we have:

Corollary 5.2. *Let G be a simply connected $2n$-dimensional nilpotent Lie group with a left invariant structure J. We suppose that the inclusion $\bigwedge^{*}\mathfrak{g}_{1,0}\otimes\bigwedge^{*}\mathfrak{g}^{*}_{0,1}\subset A^{*,*}(G/\Gamma)$ induces an isomorphism*

$$H^{*,*}_{\bar\partial}(\mathfrak{g})\cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

Let $\mu\in\bigwedge^{2}\mathfrak{g}_{1,0}$ be a holomorphic poisson bi-vector field. Then the inclusion $\bigwedge^{*}\mathfrak{g}_{1,0}\otimes\bigwedge^{*}\mathfrak{g}^{*}_{0,1}\subset A^{0,*}(G/\Gamma,\bigwedge^{*}T_{1,0}G/\Gamma)$ induces an isomorphism between the total cohomology of $(\bigwedge^{*}\mathfrak{g}_{1,0}\otimes\bigwedge^{*}\mathfrak{g}^{*}_{0,1},\bar\partial,[\mu\bullet])$ and the total cohomology of $(A^{0,*}(G/\Gamma,\bigwedge^{*}T_{1,0}G/\Gamma),\bar\partial,[\mu\bullet])$. Hence the total cohomology of $(\bigwedge^{*}\mathfrak{g}_{1,0}\otimes\bigwedge^{*}\mathfrak{g}^{*}_{0,1},\bar\partial,[\mu\bullet])$ is isomorphic to $H^{*}(G/\Gamma,\mu)$.

Proof. In the proof of Theorem 3.2, we showed that the inclusion

$$\bigwedge^{*}\mathfrak{g}_{1,0}\otimes\bigwedge^{*}\mathfrak{g}^{*}_{0,1}\subset A^{0,*}(G/\Gamma,\bigwedge^{*}T_{1,0}G/\Gamma)$$

induces an $\bar\partial$-cohomology isomorphism. Thus we can apply Lemma 5.1. □

Example 1. Let

$$N=\left\{\left(\begin{array}{ccc}1&a&c\\0&1&b\\0&0&1\end{array}\right): a,b,c\in\mathbb{C}\right\}$$

and $G = N \times \mathbb{C}$. Then G admits a lattice Γ. We take a basis X, Y, Z, W of $\mathfrak{g}_{1,0}$ such that $[X, Y] = Z$ and other brackets are 0, and a basis $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ of $\mathfrak{g}^*_{0,1}$ such that $d\bar{z} = -\bar{x} \wedge \bar{y}$ and $d\bar{x} = d\bar{y} = d\bar{w} = 0$. We consider two holomorphic poisson bi-vector fields

$$\mu_1 = X \wedge Y + Z \wedge W$$

and

$$\mu_2 = X \wedge Z + Y \wedge W.$$

Then by Corollary 5.2, we have

$$H^1(G/\Gamma, \mu_1) \cong \langle \bar{x}, \bar{y}, \bar{w}, Z, W\rangle$$

and

$$H^1(G/\Gamma, \mu_2) \cong \langle \bar{x}, \bar{y}, \bar{w}, Y, Z, W\rangle.$$

Hence we have $H^*(G/\Gamma, \mu_1) \ncong H^*(G/\Gamma, \mu_2)$.

Remark 3. μ_1 and μ_2 are induced by holomorphic symplectic forms on G/Γ. It is known that real poisson cohomology of real symplectic manifolds are only de Rham cohomology (see [26]). On the other hand, these cohomologies $H^*(G/\Gamma, \mu_1)$ and $H^*(G/\Gamma, \mu_2)$ are not isomorphic to the Dolbeault cohomology $H^*(dG^*(G/\Gamma, J))$ and we can classify holomorphic symplectic forms by using holomorphic poisson cohomology. In Section 9, we give examples of solvmanifolds with holomorphic symplectic forms such that the holomoprhic poisson cohomologies are isomorphic to the Dolbeault cohomologies.

6. DGAs and generalized deformations of solvmanifolds of splitting type

6.1. DGA.

In this section, we consider:

Assumption 6.1. *G is the semi-direct product $\mathbb{C}^n \ltimes_\phi N$ with a left-invariant complex structure $J = J_{\mathbb{C}} \oplus J_N$ so that:*
(1) N is a simply connected nilpotent Lie group with a left-invariant complex structure J_N. Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of $\mathbb{C}^n$ and N respectively.
(2) For any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of (N, J_N).
(3) ϕ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ of N.
(4) G has a lattice Γ. (Then Γ can be written by $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ such that Γ' and Γ'' are lattices of $\mathbb{C}^n$ and N respectively and for any $t \in \Gamma'$ the action $\phi(t)$ preserves Γ''.)
(5) The inclusion $\bigwedge^* \mathfrak{n}^*_{1,0} \otimes \bigwedge^* \mathfrak{n}^*_{0,1} \subset A^{*,*}(N/\Gamma'')$ induces an isomorphism

$$H^{*,*}_{\bar\partial}(\mathfrak{n}) \cong H^{*,*}_{\bar\partial}(N/\Gamma'').$$

Consider the decomposition $\mathfrak{n} \otimes \mathbb{C} = \mathfrak{n}_{1,0} \oplus \mathfrak{n}^{0,1}$. By the condition (2), this decomposition is a direct sum of $\mathbb{C}^n$-modules. By the condition (3) we have a basis $Y_1, \dots, Y_m$ of $\mathfrak{n}^{1,0}$ such that the action ϕ on $\mathfrak{n}_{1,0}$ is represented by $\phi(t) = \mathrm{diag}(\alpha_1(t), \dots, \alpha_m(t))$. Since Y_j is a left-invariant vector field on N, the vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes_\phi N$ is left-invariant. Hence we have a basis $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$ of $\mathfrak{g}_{1,0}$. Let $x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m$ be the basis of $\mathfrak{g}^*_{1,0}$ which is dual to $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$. Then we have

$$\bigwedge^p \mathfrak{g}^*_{1,0} \otimes \bigwedge^q \mathfrak{g}^*_{0,1} = \bigwedge^p \langle x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m\rangle \otimes \bigwedge^q \langle \bar{x}_1, \dots, \bar{x}_n, \bar\alpha_1^{-1} \bar{y}_1, \dots, \bar\alpha_m^{-1} \bar{y}_m\rangle.$$

Lemma 6.2. ([12, Lemma 2.2]) *Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a C^∞-character of $\mathbb{C}^n$. There exists a unique unitary character β such that $\alpha\beta^{-1}$ is holomorphic.*

By this lemma take the unique unitary characters β_i and γ_i on $\mathbb{C}^n$ such that $\alpha_i \beta_i^{-1}$ and $\bar\alpha \gamma_i^{-1}$ are holomorphic.

Theorem 6.3. ([12, Corollary 4.2]) *Let $B_\Gamma^{*,*} \subset A^{*,*}(G/\Gamma)$ be the subDBiA of $A^{*,*}(G/\Gamma)$ given by*

$$B_\Gamma^{p,q} = \left\langle x_I \wedge \alpha_J^{-1}\beta_J y_J \wedge \bar{x}_K \wedge \bar{\alpha}_L^{-1}\gamma_L \bar{y}_L \middle| \begin{array}{c} |I|+|J|=p,\, |K|+|L|=q \\ (\beta_J\gamma_L)_{|\Gamma} = 1 \end{array} \right\rangle.$$

Then the inclusion $B_\Gamma^{*,*} \subset A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism

$$H_{\bar\partial}^{*,*}(B_\Gamma^{*,*}) \cong H_{\bar\partial}^{*,*}(G/\Gamma).$$

In this paper we consider the following assumption.

Assumption 6.4. $\alpha_{[m]} = 1$ *where we write $[m] = \{1, 2, \dots, m\}$.*

In this assumption, by Theorem 3.1, we have $dy_{[m]} = 0$ and hence the holomorphic canonical bundle of G/Γ is trivialized by the global holomorphic frame $x_{[n]} \wedge y_{[m]}$. Then we have:

Theorem 6.5. *Let G be a Lie group as Assumption 6.1 with Assumption 6.4. We define the subspace*

$$C_\Gamma^{p,q} = \left\langle X_I \wedge \alpha_J \beta_J^{-1} Y_J \otimes \bar{x}_K \wedge \bar{\alpha}_L^{-1}\gamma_L \bar{y}_L \middle| \begin{array}{c} |I|+|J|=p,\, |K|+|L|=q \\ (\beta_J^{-1}\gamma_L)_{|\Gamma} = 1 \end{array} \right\rangle$$

of $A^{0,q}(G/\Gamma, \bigwedge^p T^{1,0}G/\Gamma)$. We denote $C^k = \oplus_{p+q=k} C^{p,q}$. Then $(C^*, \bar\partial)$ is a subDGA of $dG^*(G/\Gamma, J)$ and the inclusion $C^* \subset dG^*(G/\Gamma, J)$ induces a cohomology isomorphism.

Proof. We consider the weight decomposition

$$\bigwedge(\mathfrak{n}_{1,0} \oplus \mathfrak{n}_{0,1}^*) = \bigoplus V_{\epsilon_i}$$

of $\mathbb{C}^n$-action via ϕ Since $\phi(t)$ induces a semi-simple automorphism on the DGA $\bigwedge(\mathfrak{n}_{1,0} \oplus \mathfrak{n}_{0,1}^*)$ for any $t \in \mathbb{C}^n$, we have $V_{\epsilon_i} \wedge V_{\epsilon_j} \subset V_{\epsilon_i\epsilon_j}$, $[V_{\epsilon_i} \bullet V_{\epsilon_j}] \subset V_{\epsilon_i\epsilon_j}$ and $\bar\partial(V_{\epsilon_i}) \subset V_{\epsilon_i}$. Taking the unitary character ζ_i of $\mathbb{C}^n$ such that $\epsilon_i\zeta_i^{-1}$ is holomorphic as Lemma 6.2, we have

$$C_\Gamma^* = \bigoplus_{(\zeta_i)_{|\Gamma}=1} \bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\zeta_i^{-1} V_{\epsilon_i}.$$

Hence C_Γ^* is closed under wedge product. Since $\epsilon_i\zeta_i^{-1}$ is holomorphic, we have

$$\begin{aligned}&[\bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\zeta_i^{-1} V_{\epsilon_i} \bullet \bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_j\zeta_j^{-1} V_{\epsilon_j}] \\ &\qquad\qquad \subset \bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\epsilon_j\zeta_i^{-1}\zeta_j^{-1} V_{\epsilon_i\epsilon_j}\end{aligned}$$

and

$$\begin{aligned}&\bar\partial\left(\bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\zeta_i^{-1} V_{\epsilon_i}\right) \\ &\qquad = \bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\zeta_i^{-1} \bar\partial V_{\epsilon_i} \subset \bigwedge((\mathbb{C}^n)_{1,0} \oplus (\mathbb{C}^n)_{0,1}^*) \otimes \epsilon_i\zeta_i^{-1} V_{\epsilon_i}.\end{aligned}$$

Hence C_Γ^* is a subDGA of $dG^*(G/\Gamma, J)$.

We will show that $C_\Gamma^* \subset dG^*(G/\Gamma, J)$ induces a cohomology isomorphism. Since the holomorphic canonical bundle of G/Γ is trivialized by the global holomorphic frame $x_{[n]} \wedge y_{[m]}$, we have the isomorphism

$$\bigwedge^p T_{1,0}G/\Gamma \cong \bigwedge^{n+m-p} T_{1,0}^*G/\Gamma$$

which is given by

$$\bigwedge^p \mathfrak{g}_{1,0} \ni X_I \wedge \alpha_J Y_J \mapsto x_I \wedge \alpha_{[m]-J}^{-1} y_J \in \bigwedge^{n+m-p} \mathfrak{g}_{1,0}^*.$$

By this isomorphism we have the commutative diagram

$$\begin{tikzcd} B_{\Gamma}^{n+m-p,*} \arrow[r] \arrow[d, "\cong"] & A^{n+m-p,*}(G/\Gamma) \arrow[d, "\cong"] \\ C_{\Gamma}^{p,*} \arrow[r] & A^{0,*}(G/\Gamma, \bigwedge^p T_{1,0}G/\Gamma). \end{tikzcd}$$

Hence by Theorem 6.3, $C_{\Gamma}^{p,*}$ is a subcomplex of $A^{0,*}(G/\Gamma, \bigwedge^p T_{1,0}G/\Gamma)$ and the inclusion $C_{\Gamma}^{p,*} \subset A^{0,*}(G/\Gamma, \bigwedge^p T_{1,0}G/\Gamma)$ induces a cohomology isomorphism for each p and this implies the theorem. $\square$

Remark 4. Consider the following condition:

($\mathcal{C}_r$) For $r \in \mathbb{N}$, for any $J, L \subset [m]$ with $|J| + |L| \leq r$, $(\beta_J^{-1}\gamma_L)_{|\Gamma} = 1$ if and only if $(\alpha_J \bar{\alpha}_L^{-1})_{|\Gamma} = 1$.

If $(\alpha_J \alpha_L^{-1})_{|\Gamma} = 1$, then $\alpha_J\alpha_L^{-1}$ is unitary and we have $\alpha_J\alpha_L^{-1} = \beta_J\beta_L^{-1}$. By this if the condition $(\mathcal{C}_r)$ holds, then for $p, q \in N$ with $p + q \leq r$, we have

$$C_{\Gamma}^{p,q} = \left\langle X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L | (\alpha_J\alpha_L^{-1})_{|\Gamma} = 1 \right\rangle.$$

Hence we have an embedding $C_{\Gamma}^{s} \subset dG^s(\mathbb{C}^n \oplus \mathfrak{n}, J)$ for any $s \leq r$.

6.2. **Holomorphic poisson cohomology.** We consider the cohomology of holomorphic poisson solvmanifolds.

Corollary 6.6. *Let G be a Lie group as Assumption 6.1 with Assumption 6.4. We consider the DBiA $(C_{\Gamma}^{*,*}, \bar{\partial})$ as Theorem 6.5. Let $\mu \in C_{\Gamma}^{2.0}$ be a holomorphic poisson bi-vector field. Then the inclusion $C_{\Gamma}^{*,*} \subset A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma)$ induces an isomorphism between the total cohomology of the double complex $(C_{\Gamma}^{*,*}, \bar{\partial}, [\mu\bullet])$ and the total cohomology of the double complex $(A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma), \bar{\partial}, [\mu\bullet])$. Hence total cohomology of the double complex $(C_{\Gamma}^{*,*}, \bar{\partial}, [\mu\bullet])$ is isomorphic to $H(G/\Gamma, \mu)$.*

Proof. In the proof of Theorem 6.5, we showed that the inclusion $C_{\Gamma}^{*,*} \subset A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma)$ induces a $\bar{\partial}$-cohomology isomorphism. Thus we can apply Lemma 5.1. $\square$

6.3. **Generalized deformation.** Let G be a Lie group as Assumption 6.1 with Assumption 6.4. We consider the left-invariant Hermitian metric

$$g = \sum x_i\bar{x}_i + \sum \alpha_i^{-1}\bar{\alpha}_i^{-1} y_i\bar{y}_i.$$

Then the $\mathbb{C}$-anti-linear Hodge star operator

$$\bar{*}_g : A^{0,q}(G/\Gamma, \bigwedge^{p} T_{1,0}G/\Gamma) \to A^{0,n+m-q}(G/\Gamma, \bigwedge^{n+m-p} T_{1,0}G/\Gamma)$$

is given by

$$\bar{*}_g(X_I \wedge \alpha_J Y_J \otimes \bar{x}_K \wedge \bar{\alpha}_L^{-1}\bar{y}_L) = X_{[n]-I} \wedge \alpha_{[m]-J} Y_{[m]-J} \otimes \bar{x}_{[n]-K} \wedge \bar{\alpha}_{[m]-L}^{-1}\bar{y}_{[m]-L}.$$

Since we have $\alpha_{[m]-J} = \alpha_J^{-1}$ and $\beta_{[m]-J} = \beta_J^{-1}$, we have

$$\begin{aligned} &\bar{\star}(X_I \wedge \alpha_J\beta_J^{-1} Y_J \otimes \bar{x}_K \wedge \bar{\alpha}_L^{-1}\gamma_L\bar{y}_L) \\ &\qquad = X_{[n]-I} \wedge \alpha_{[m]-J}\beta_J Y_{[m]-J} \otimes \bar{x}_{[n]-K} \wedge \bar{\alpha}_{[m]-L}^{-1}\gamma_L^{-1}\bar{y}_{[m]-L} \\ &\qquad = X_{[n]-I} \wedge \alpha_{[m]-J}\beta_{[m]-J}^{-1} Y_{[m]-J} \otimes \bar{x}_{[n]-K} \wedge \bar{\alpha}_{[m]-L}^{-1}\gamma_{[m]-L}\bar{y}_{[m]-L} \in C^*. \end{aligned}$$

Hence we have $\bar{*}(C^*) \subset C^*$. For the above $X_1, \dots X_n, Y_1, \dots, Y_m$, $x_1, \dots x_n, y_1, \dots, y_m$, the complex structure on G is

$$J = \sqrt{-1}\sum(X_1 \otimes x_i + Y_i \otimes y_i) - \sqrt{-1}\sum(\bar{X}_i \otimes \bar{x}_i + \bar{Y}_i \otimes \bar{y}_i).$$

and this gives the generalized complex structure

$$\mathcal{J} = \sqrt{-1}\sum(X_i \otimes x_i + Y_i \otimes y_i + \bar{x}_i \otimes \bar{X}_i + \bar{y}_i \otimes \bar{Y}_i) - \sqrt{-1}\sum(\bar{X}_i \otimes \bar{x}_i + \bar{Y}_i \otimes \bar{y}_i + x_i \otimes X_i + y_i \otimes Y_i).$$

By Theorem 6.5, Theorem 4.1 and Lemma 4.2, we have:

Corollary 6.7. *Any deformation of the generalized complex structure $\mathcal{J}$ is equivalent to*

$$\mathcal{J} = \sqrt{-1}\sum(X_i^\epsilon \otimes x_i^\epsilon + Y_i^\epsilon \otimes y_i^\epsilon + (\bar{x}_i)^\epsilon \otimes (\bar{X}_i)^\epsilon + (\bar{y}_i)^\epsilon \otimes (\bar{Y}_i)^\epsilon)$$
$$- \sqrt{-1}\sum((\bar{X}_i)^{\bar{\epsilon}} \otimes (\bar{x}_i)^{\bar{\epsilon}} + (\bar{Y}_i)^{\bar{\epsilon}} \otimes (\bar{y}_i)^{\bar{\epsilon}} + x_i^{\bar{\epsilon}} \otimes X_i^{\bar{\epsilon}} + y_i^{\bar{\epsilon}} \otimes Y_i^{\bar{\epsilon}}).$$

for some $\epsilon \in C^2_\Gamma$ where we write $E^\epsilon = E + i_E\epsilon$ for $E \in L_{\mathcal{J}}$.

We consider the group $\mathrm{Aut}(N) \ltimes (\mathbb{C}^n \times N)$ of transformations. Then by $\mathbb{C}^n \ltimes_\phi N \ni (g, x) \mapsto (\phi(g), g, x) \in \mathrm{Aut}(N) \ltimes (\mathbb{C}^n \times N)$, we have the inclusion $G \subset \mathrm{Aut}(N) \ltimes (\mathbb{C}^n \times N)$. Hence we can represent $G/\Gamma = \Gamma\backslash(\mathbb{C}^n \times N)$. In this representation, the DGA $dG^*(G/\Gamma, J)$ is represented by the space $dG^*(\mathbb{C}^n \times N, J)^\Gamma$ which consists of the elements of $dG^*(\mathbb{C}^n \times N, J)$ which fixed by the Γ-action.

We assume the condition $(\mathcal{C}_2)$ in Remark 4 holds. By

$$C^{p,q}_\Gamma = \left\langle X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L | (\alpha_J \bar{\alpha}_L^{-1})_{|\Gamma} = 1 \right\rangle$$

for any $p, q \in \mathbb{N}$ with $p + q \leq 2$, considering the subDGA $dG^*(\mathbb{C}^n \oplus \mathfrak{n}, J)$ of $dG^*(\mathbb{C}^n \times N, J)$, we have

$$C^2_\Gamma \subset dG^*(G/\Gamma, J) \cap dG^*(\mathbb{C}^n \oplus \mathfrak{n}, J).$$

By Corollary 7.4, we have:

Theorem 6.8. *We assume the condition $(\mathcal{C}_2)$ in Remark 4 holds. As we represent $G/\Gamma = \Gamma\backslash(\mathbb{C}^n \times N)$, any small deformation of the generalized complex structure $\mathcal{J}$ is equivalent to a generalized complex structure on $\Gamma\backslash(\mathbb{C}^n \times N)$ induced by a $(\mathbb{C}^n \times N)$-left-invariant generalized complex structure on $\mathbb{C}^n \times N$ which is fixed by the Γ-action.*

Example 2. Let $G = \mathbb{C} \ltimes_\phi N$ such that

$$N = \left\{ \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : z, v, w \in \mathbb{C} \right\}$$

and ϕ is given by

$$\phi(s + \sqrt{-1}t) \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-2\pi\sqrt{-1}t}\bar{z} & \frac{1}{2}e^{-4\pi\sqrt{-1}t}\bar{z}^2 & e^{-2\pi\sqrt{-1}t}w \\ 0 & 1 & e^{-2\pi\sqrt{-1}t}\bar{z} & v \\ 0 & 0 & 1 & e^{2\pi\sqrt{-1}t}z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have a lattice

$$\Gamma = (\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$$

such that

$$\Gamma'' = \left\{ \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : z, v, w \in (\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \right\}.$$

Then $\phi_{|\Gamma}$ is trivial. Hence the presentation $\Gamma\backslash(\mathbb{C}\times N)$ as above is identified with the nilmanifold $(\mathbb{C}\times N)/\Gamma$. In this case, Theorem 6.8 says Theorem 4.3.

We suppose the condition $(\mathcal{C}_r)$ holds for any $r \in N$. Then we have an embedding $C^* \subset dG^*(\mathbb{C}^n \oplus \mathfrak{n}, J)$. Consider the Lie algebra $\mathbb{C}^n \oplus \mathfrak{n}$ with the Hermitian metric

$$h = \sum x_i \bar{x}_i + \sum y_i \bar{y}_i$$

and the $\mathbb{C}$-anti-linear Hodge star operator

$$\bar{*}_h : \bigwedge^p (\mathbb{C}^n \oplus \mathfrak{n})_{1,0} \otimes \bigwedge^q (\mathbb{C}^n \oplus \mathfrak{n})^*_{0,1} \to \bigwedge^{n+m-p} (\mathbb{C}^n \oplus \mathfrak{n})_{1,0} \otimes \bigwedge^{n+m-q} (\mathbb{C}^n \oplus \mathfrak{n})^*_{0,1}.$$

For $C^{p,q}_\Gamma \ni X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L$,

$$\bar{*}_g(X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L) = X_{[n]-I} \wedge Y_{[m]-J} \otimes \bar{x}_{[n]-K} \wedge \bar{y}_{[m]-L} = \bar{*}_h(X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L).$$

Consider the nilmanifold $(\mathbb{C}^n \times N)/(\Gamma' \times \Gamma'')$ with the left-invariant Hermitian metric h. Then we have $\operatorname{Ker} \Box_{h|_{dG^*(\mathbb{C}^n\oplus\mathfrak{n},J)}} = \mathcal{H}^*_h((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma''), J)$. We we have the commutative diagram of inclusions

$$\begin{tikzcd} C^*_\Gamma \arrow[r, "\subset"] & dG^*(\mathbb{C}^n \oplus \mathfrak{n}, J) \\ \mathcal{H}^*(G/\Gamma, J) \arrow[u, "\subset"] \arrow[r, "\subset"] & \mathcal{H}^*_h((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma''), J) \arrow[u, "\subset"'] \end{tikzcd}$$

By this diagram we can take a basis $\eta_1, \dots, \eta_l, \dots, \eta_k$ of $\mathcal{H}^*_h((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma''), J)$ such that $\eta_1, \dots, \eta_l$ is a basis of $\mathcal{H}^*(G/\Gamma, J)$. For parameters $t = (t_1, \dots, t_k)$, we consider the formal power series $\phi(t) = \phi(t_1, \dots, t_k)$ for the constrution of $Kur^{gen}((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma''), J)$ by the basis $\eta_1, \dots, \eta_l, \dots, \eta_k$ as Section 4.1. By the inclusion $C^*_\Gamma \subset dG^*(\mathbb{C}^n \oplus \mathfrak{n}, J)$ of DGAs, for parameters $t = (t_1, \dots, t_l)$ the restriction $\phi(t_1, \dots, t_k, 0, \dots, 0)$ gives us the construction of $Kur^{gen}(G/\Gamma, J)$, that is to say:

Theorem 6.9. *We suppose the condition $(\mathcal{C}_r)$ holds for any $r \in N$. We have an analytic embedding of $Kur^{gen}(G/\Gamma, J)$ into $Kur^{gen}((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma''), J)$ as*

$$\begin{tikzcd} Kur^{gen}(G/\Gamma, J) \arrow[r, equal] \arrow[d] & \{t = (t_1, \dots, t_l, 0, \dots, 0) \mid |t| < \delta,\ H[\phi(t) \bullet \phi(t)] = 0\} \arrow[d, "\subset"] \\ Kur^{gen}((\mathbb{C}^n \times N)/(\Gamma' \times \Gamma'') \arrow[r, equal] & \{t = (t_1, \dots, t_k) \mid |t| < \delta,\ H[\phi(t) \bullet \phi(t)] = 0\}. \end{tikzcd}$$

By theorem 4.4, we have:

Corollary 6.10. *We suppose the condition $(\mathcal{C}_r)$ holds for any $r \in N$. Suppose (N, J_N) is a complex ν-step nilpotent Lie group. Then $\delta = \infty$ and $Kur^{gen}(G/\Gamma, J)$ is cut out by polynomial equations of degree at most ν.*

Remark 5. We consider the following condition:

($\mathcal{D}_r$) For $r \in \mathbb{N}$, for any $J, L \subset [m]$ with $|J|+|L| \le r$, $(\beta_J^{-1}\gamma_L)_{|\Gamma} = 1$ if and only if $\alpha_J \bar{\alpha}_L^{-1} = 1$.

If the condition $\mathcal{D}_r$ holds, then for $p, q \in N$ with $p+q \le r$ we have

$$C_\Gamma^{p,q} = \left\langle X_I \wedge Y_J \otimes \bar{x}_K \wedge \bar{y}_L | \alpha_J \alpha_L^{-1} = 1 \right\rangle.$$

Hence we have $C_\Gamma^* \subset dG^*(\mathfrak{g}, J)$. This implies that $\phi(u) \in Kur^{ext}(G/\Gamma)$ gives a left-invariant generalized complex structure on G/Γ. We have:

Theorem 6.11. *Suppose that the condition ($\mathcal{D}_2$) holds. Then any deformation of the generalized complex structure $\mathcal{J}$ is equivalent to a generalized complex structure on G/Γ which is induced by a left-invariant generalized complex structure on G.*

7. DGAs of complex parallelizable solvmanifolds

7.1. **DGA.** Let G be a simply connected n-dimensional complex solvable Lie group. Denote by $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) the Lie algebra of the left-invariant holomorphic (anti-holomorphic) vector fields on G. As a Lie algebra we have an isomorphism $\mathfrak{g}_+ \cong \mathfrak{g}_-$ by the complex complex conjugate. Let N be the nilradical of G. We can take a simply connected complex nilpotent subgroup $C \subset G$ such that $G = C \cdot N$ (see [6]). Since C is nilpotent, the map

$$C \ni c \mapsto (\mathrm{Ad}_c)_s \in \mathrm{Aut}(\mathfrak{g}_+)$$

is a homomorphism where $(\mathrm{Ad}_c)_s$ is the semi-simple part of Ad_s. Denote by $\bigwedge \mathfrak{g}_+^*$ (resp. $\bigwedge \mathfrak{g}_-^*$) the sub-DGA of $(A^{*,0}, \partial)$ (resp. $(A^{0,*}, \bar{\partial})$ which consists of the left-invariant holomorphic (anti-holomorphic) forms. As a DGA, we have an isomorphism between $(\bigwedge \mathfrak{g}_+^*, \partial)$ and $(\bigwedge \mathfrak{g}_-^*, \bar{\partial}))$ given by the complex conjugate.

We have a basis $X_1, \dots, X_n$ of $\mathfrak{g}_+$ such that $(\mathrm{Ad}_c)_s = \mathrm{diag}(\alpha_1(c), \dots, \alpha_n(c))$ for $c \in C$. Let $x_1, \dots, x_n$ be the basis of $\mathfrak{g}_+^*$ which is dual to $X_1, \dots, X_n$.

Theorem 7.1. ([13, Corollary 6.2 and its proof]) *Let B_Γ^* be the subcomplex of $(A^{0,*}(G/\Gamma), \bar{\partial})$ defined as*

$$B_\Gamma^* = \left\langle \frac{\bar{\alpha}_I}{\alpha_I}\bar{x}_I \middle| \left(\frac{\bar{\alpha}_I}{\alpha_I}\right)_{|\Gamma} = 1 \right\rangle.$$

Then the inclusion $B_\Gamma^* \subset A^{0,*}(G/\Gamma)$ induces an isomorphism

$$H^*(B_\Gamma^*) \cong H^{0,*}(G/\Gamma).$$

By this theorem we prove:

Theorem 7.2. *We consider the subspace*

$$C_\Gamma^{p,q} = \bigwedge^p \mathfrak{g}_+ \otimes B_\Gamma^q$$

of $A^{0,q}(G/\Gamma, \bigwedge^p T_{1,0}G/\Gamma)$. Denote $C_\Gamma^r = \bigoplus_{p+q=r} C_\Gamma^{p,q}$. Then C_Γ^* is a subDGA of $dG^*(G/\Gamma, J)$ and the inclusion induces a cohomology isomorphism.

Proof. By Theorem 7.1, the inclusion $C_\Gamma^{*,*} \subset A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma)$ induces a cohomology isomorphism

$$\bigwedge^* \mathfrak{g}_+ \otimes H^*(B_\Gamma^*) \cong \bigwedge^* \mathfrak{g}_+ \otimes H^{0,*}(G/\Gamma).$$

Hence it is sufficient to show that C_Γ^* is closed under the Schouten bracket.

For any $X \in \mathfrak{g}_+$, we have

$$X(\alpha_I^{-1}) = \alpha_I^{-1}\alpha_I d\alpha_I^{-1}(X).$$

Since $\alpha_I d\alpha_I^{-1}$ is left-invariant, $\alpha_I^{-1}\alpha_I d\alpha_I^{-1}(X)$ is constant. Hence for some constant c, we have

$$X(\alpha_I^{-1}) = c\alpha_I^{-1}.$$

Since $\bar{\alpha}_I$ and $\bar{x}_I \in \bigwedge \mathfrak{g}_-^*$ are anti-holomorphic, we have

$$L_X\left(\frac{\bar{\alpha}_I}{\alpha_I}\bar{x}_I\right) = c\frac{\bar{\alpha}_I}{\alpha_I}\bar{x}_I.$$

Hence $\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*$ is closed under the Schouten bracket. $\square$

7.2. **Holomorphic poisson cohomology.** Since the incluion $C_\Gamma^{*,*} \subset A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma)$ induces a $\bar{\partial}$-cohomology isomorphism, by Lemma 5.1, we have:

Theorem 7.3. *Let $\mu \in C_\Gamma^{2,0}$ be a holomorphic poisson bi-vector field. Then the inclusion $C_\Gamma^{*,*} \subset A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma)$ induces an isomorphism between the total cohomology of the double complex $(C_\Gamma^{*,*}, \bar{\partial}, [\mu\bullet])$ and the total cohomology of the double complex $(A^{0,*}(G/\Gamma, \bigwedge^* T_{1,0}G/\Gamma), \bar{\partial}, [\mu\bullet])$. Hence total cohomology of the double complex $(C_\Gamma^{*,*}, \bar{\partial}, [\mu\bullet])$ is isomorphic to $H(G/\Gamma, \mu)$.*

7.3. **Generalized deformation.** We consider the Hermitian metric

$$g = \sum x_i\bar{x}_i.$$

Then for $x_I\frac{\bar{\alpha}_J}{\alpha_J}\bar{x}_J \in \bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*$, since G is unimodulor, we have

$$\bar{*}_g(x_I \wedge \frac{\bar{\alpha}_J}{\alpha_J}\bar{x}_J) = x_{I'} \wedge \frac{\alpha_J}{\bar{\alpha}_J}\bar{x}_{J'} = x_{I'} \wedge \frac{\bar{\alpha}_{J'}}{\alpha_{J'}}\bar{x}_{J'} \in \bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*$$

where I' and J' are complements of I and J respectively. Hence we have $\bar{*}_g(\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*) \subset \bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*$. For the basis $X_1, \dots X_n$, $x_1, \dots x_n$, the complex structure on G is

$$J = \sqrt{-1}\sum(X_1 \otimes x_i) - \sqrt{-1}\sum(\bar{X}_i \otimes \bar{x}_i).$$

and this gives the generalized complex structure

$$\mathcal{J} = \sqrt{-1}\sum(X_i \otimes x_i + \bar{x}_i \otimes \bar{X}_i) - \sqrt{-1}\sum(\bar{X}_i \otimes \bar{x}_i + x_i \otimes X_i).$$

By theorem 7.2, as the last section, we have:

Corollary 7.4. *Any deformation of the generalized complex structure $\mathcal{J}$ is equivalent to*

$$\mathcal{J} = \sqrt{-1}\sum(X_i^\epsilon \otimes x_i^\epsilon + (\bar{x}_i)^\epsilon \otimes (\bar{X}_i)^\epsilon) - \sqrt{-1}\sum((\bar{X}_i)^{\bar{\epsilon}} \otimes (\bar{x}_i)^{\bar{\epsilon}} + x_i^{\bar{\epsilon}} \otimes X_i^{\bar{\epsilon}}).$$

for some $\epsilon \in C_\Gamma^2$ where we write $E^\epsilon = E + i_E\epsilon$ for $E \in L_{\mathcal{J}}$.

We consider the following condition:

($\mathcal{E}_r$) For $r \in \mathbb{N}$, for any $I \subset [n]$ with $|I| \le r$, $\left(\frac{\bar{\alpha}_I}{\alpha_I}\right)_{|\Gamma} = 1$ if and only if $\alpha_I = 1$.

If the condition ($\mathcal{E}_r$) holds, then for $p \le r$, we have

$$B_\Gamma^q = \langle \bar{x}_I | \alpha_I = 1,\ |I| = q\rangle$$

and hence we have $B_\Gamma^p \subset \bigwedge^p \mathfrak{g}_-^*$. Thus we have:

Theorem 7.5. *Suppose that the condition ($\mathcal{E}_2$) holds. Then any deformation of the generalized complex structure $\mathcal{J}$ is equivalent to a generalized complex structure on G/Γ which is induced by a left-invariant generalized complex structure on G.*

Example 3. Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Then we have $a + \sqrt{-1}b, c + \sqrt{-1}d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}b)$ and $\phi(c + \sqrt{-1}d)$ are conjugate to elements of $SL(4, \mathbb{Z})$ where we regard $SL(2, \mathbb{C}) \subset SL(4, \mathbb{R})$ (see [10]). Hence we have a lattice $\Gamma = (\mathbb{Z}(a+\sqrt{-1}b)+\mathbb{Z}(c+\sqrt{-1}d)) \ltimes_\phi \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$.

We take $C = \mathbb{C}$. For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes_\phi \mathbb{C}^2$, we have the basis $\frac{\partial}{\partial z_1}, e^{z_1}\frac{\partial}{\partial z_2}, e^{-z_1}\frac{\partial}{\partial z_3}$ of $\mathfrak{g}_+$ such that $(\mathrm{Ad}_{(z_1,)})_s = \mathrm{diag}(1, e^{z_1}, e^{-z_1})$.

If $b \notin \pi\mathbb{Z}$ or $c \notin \pi\mathbb{Z}$, then the condition $(\mathcal{E}_2)$ holds and we have

$$B^1_\Gamma = \langle d\bar{z}_1 \rangle,$$
$$B^2_\Gamma = \langle d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$
$$B^3_\Gamma = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.$$

If $b \in \pi\mathbb{Z}$ and $c \in \pi\mathbb{Z}$, then $(\mathcal{E}_2)$ does not hold. It is known that there exists a small deformation of J which can not be induced by a G-left-invariant complex structure on G (see [19], [10]).

8. DGAs of symplectic solvmanifolds

Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. There exists a subvector space (not necessarily Lie algebra) V of $\mathfrak{g}$ so that $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of vector spaces and for any $A, B \in V$ $(\mathrm{ad}_A)_s(B) = 0$ where $(ad_A)_s$ is the semi-simple part of ad_A (see [7, Proposition III.1.1]). We define the map $\mathrm{ad}_s : \mathfrak{g} \to D(\mathfrak{g})$ as $\mathrm{ad}_{sA+X} = (\mathrm{ad}_A)_s$ for $A \in V$ and $X \in \mathfrak{n}$. Then we have $[\mathrm{ad}_s(\mathfrak{g}), \mathrm{ad}_s(\mathfrak{g})] = 0$ and ad_s is linear (see [7, Proposition III.1.1]). Since we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$, the map $\mathrm{ad}_s : \mathfrak{g} \to D(\mathfrak{g})$ is a representation and the image $\mathrm{ad}_s(\mathfrak{g})$ is abelian and consists of semi-simple elements. We denote by $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ the extension of ad_s. Then $\mathrm{Ad}_s(G)$ is diagonalizable.

Let T be the Zariski-closure of $\mathrm{Ad_s}(G)$ in $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Denote by $\mathrm{Char}(T)$ the set of all 1-dimensional algebraic group representations of T. Set

$$\mathcal{C}_\Gamma := \left\{\beta \circ \mathrm{Ad}_s \in \mathrm{Hom}\,(G; \mathbb{C}^*) \,|\beta \in \mathrm{Char}(T),\ (\beta \circ \mathrm{Ad}_s)_{|\Gamma} = 1\right\}.$$

We consider the DGrA

$$\bigoplus_{\alpha \in \mathcal{C}_\Gamma} \alpha \cdot \bigwedge \mathfrak{g}^*_\mathbb{C}$$

of $A^*(G/\Gamma)$. By $\mathrm{Ad_s}(G) \subseteq \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ we have the $\mathrm{Ad_s}(G)$-action on the DGrA $\bigoplus_{\alpha \in \mathcal{C}_\Gamma} \alpha \cdot \bigwedge \mathfrak{g}^*_\mathbb{C}$. We denote by A^*_Γ the space consisting of the $\mathrm{Ad_s}(G)$-invariant elements of $\bigoplus_{\alpha \in \mathcal{C}_\Gamma} \alpha \cdot \bigwedge \mathfrak{g}^*_\mathbb{C}$. Now we consider the inclusion $A^*_\Gamma \subset A^*(G/\Gamma)$ of

Theorem 8.1. ([11, Corollary 7.6]) *The inclusion*

$$A^*_\Gamma \subset A^*(G/\Gamma)$$

induces a cohomology isomorphism.

Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}_\mathbb{C}$ such that Ad_s is represented by diagonal matrices. Then we have $\mathrm{Ad}_{sg} X_i = \alpha_i(g) X_i$ for characters α_i of G. Let $x_1, \ldots, x_n$ be the dual basis of $X_1, \ldots, X_n$. Then the sub-DGrA A^*_Γ of the de Rham complex $A^*_\mathbb{C}(G/\Gamma)$ is given by

$$A^p_\Gamma = \left\langle \alpha_I x_I \,\middle|\, \begin{array}{c} I \subset [n], \\ (\alpha_I)_{|\Gamma} = 1 \end{array} \right\rangle.$$

We prove:

Theorem 8.2. *Let ω be a symplectic structure on G/Γ. We assume $\omega \in A^2_\Gamma$. Then A^*_Γ is a subDGA of $dG^*(G/\Gamma, \omega)$ and the inclusion $A^*_\Gamma \subset A^*_{\mathbb{C}}(G/\Gamma)$ induces a cohomology isomorphism.*

Proof. By Theorem 8.1, it is sufficient to show that A^*_Γ is closed under the bracket $[\bullet]_\omega$

We define the subspace $A^* \subset A^*_{\mathbb{C}}(G)$ as

$$A^* = \bigwedge \langle \alpha_1 x_1, \dots, \alpha_n x_n \rangle.$$

Since we have $A^*_{\mathbb{C}}(G/\Gamma) = A^*_{\mathbb{C}}(G)^\Gamma$, we have $A^*_\Gamma = A^* \cap A^*_{\mathbb{C}}(G)^\Gamma$ where $A^*_{\mathbb{C}}(G)^\Gamma$ is the space of Γ-invariant differential forms. Then A^* is a subcomplex of $A^*_{\mathbb{C}}(G)$ and we have an isomorphism $(A^*, d) \cong (\bigwedge \mathfrak{u}^*_G, d)$ for some nilpotent Lie algebra $\mathfrak{u}_G$ (see [11, Proof of Lemma 5.2]).

We define the subspace $D^* \subset C^\infty(G, \bigwedge TG) \otimes \mathbb{C}$ as

$$D^* = \bigwedge \langle \alpha_1^{-1} X_1, \dots, \alpha_n^{-1} X_n \rangle$$

We denote $D^*_\Gamma = D^* \cap (C^\infty(G, \bigwedge TG))^\Gamma$. Then D^* is dual to A^* and so D^* with the Schouten bracket is isomorphic to the Gerstenhaber algebra $\bigwedge \mathfrak{u}_G$. By this D^*_Γ is closed under the Schouten bracket.

Let $\omega \in A^*$ be a symplectic form. Then ω is a symplectic bi-linear form on the linear space $\langle \alpha_1^{-1} X_1, \dots, \alpha_n^{-1} X_n \rangle$. Hence we have the map $\omega : D^* \to A^*$. Since D^* is closed under the Schouten bracket, A^* is closed under the bracket $[\bullet]_\omega$.

Suppose $\omega \in A^*_{\mathbb{C}}(G/\Gamma) = A^*_{\mathbb{C}}(G)^\Gamma$. Then we have $\omega : (C^\infty(G, \bigwedge TG))^\Gamma \to A^*_{\mathbb{C}}(G)^\Gamma$ and hence we have $\omega : D^*_\Gamma \to A^*_\Gamma$. Since D^*_Γ is closed under the Schouten bracket, the theorem follows. □

9. Formality and weak mirror symmetry

Corollary 9.1. *Let $G = \mathbb{C}^n \ltimes_\phi \mathbb{C}^m$ with a semi-simple unimodular action $\phi : \mathbb{C}^n \to GL_m(\mathbb{C})$ (not necessarily holomorphic). We suppose G has a lattice Γ. Consider the Hermitian metric $g = \sum dz_i d\bar{z}_i + \sum \alpha_i^{-1} \bar{\alpha}_i^{-1} dw_i d\bar{w}_i$. Then $\mathcal{H}^*(G/\Gamma, J)$ is a sub-DGA of $dG^*(G/\Gamma, J)$. In particular, $\delta = \infty$ and $Kur^{gen}(G/\Gamma, J)$ is cut out by polynomial equations of degree at most 2.*

Proof. We have a coordinate $z_1, \dots, z_n, w_1, \dots, w_m$ of $\mathbb{C}^n \ltimes_\phi \mathbb{C}^m$ such that

$$\phi(z_1, \dots, z_n)(w_1, \dots, w_m) = (\alpha_1 w_1, \dots, \alpha_m w_m)$$

where α_i are C^∞-characters of $\mathbb{C}^n$.

We take unitary characters β_i and γ_i such that $\alpha_i \beta_i^{-1}$ and $\bar{\alpha} \gamma_i^{-1}$ are holomorphic. We define the subspace

$$C^{p,q}_\Gamma = \left\langle \frac{\partial}{\partial z_I} \wedge \alpha_J \beta_J^{-1} \frac{\partial}{\partial w_J} \otimes d\bar{z}_K \wedge \bar{\alpha}_L^{-1} \gamma_L d\bar{w}_L \middle| \begin{array}{c} |I| + |K| = p, \ |J| + |L| = q \\ (\beta_J \gamma_L)_{|\Gamma} = 1 \end{array} \right\rangle$$

of $A^{0,q}(G/\Gamma, \bigwedge^p T^{1,0} G/\Gamma)$. We denote $C^k_\Gamma = \oplus_{p+q=k} C^{p,q}_\Gamma$. Then C^k_Γ is a sub-DGA of $dG^*(G/\Gamma, J)$ and the inclusion $C^*_\Gamma \subset A^*$ induces a cohomology isomorphism. The operator $\bar{\partial}$ on C^*_Γ is 0.

For the $\mathbb{C}$-anti-linear Hodge star operator

$$\bar{*} : A^{0,q}(G/\Gamma, \bigwedge^p T^{1,0} G/\Gamma) \to A^{0,n+m-q}(G/\Gamma, \bigwedge^{n+m-p} T^{1,0} G/\Gamma)$$

we have

$$\bar{*}(\frac{\partial}{\partial z_I}\wedge\alpha_J\beta_J^{-1}\frac{\partial}{\partial w_J}\otimes d\bar{z}_K\wedge\bar{\alpha}_L^{-1}\gamma_L d\bar{w}_L)$$
$$=\frac{\partial}{\partial z_{[n]-I}}\wedge\alpha_{[m]-J}\bar{\beta}_J^{-1}\frac{\partial}{\partial w_{[m]-J}}\otimes dz_{[n]}\wedge dw_{[m]}\wedge d\bar{z}_{[n]-K}\wedge\bar{\alpha}^{-1}_{[m]-L}\bar{\gamma}_L d\bar{w}_{[m]-L}$$

Since we have $\alpha_{[m]-J}=\alpha_J^{-1}$ by unimodularity we have

$$\bar{\partial}(\bar{*}(\frac{\partial}{\partial z_I}\wedge\alpha_J\beta_J^{-1}\frac{\partial}{\partial w_J}\otimes d\bar{z}_K\wedge\bar{\alpha}_L^{-1}\gamma_L d\bar{w}_L))=0.$$

Hence we have $\bar{*}\bar{\partial}\bar{*}(C^*_\Gamma)=0$ and the corollary follows. $\square$

We consider

$$A^*_\Gamma=\left\langle dz_I\wedge dw_J\wedge d\bar{z}_K\wedge d\bar{w}_L\middle|(\alpha_J\bar{\alpha}_L)_{|\Gamma}=1\right\rangle.$$

Suppose G/Γ has a symplectic form ω such that $\omega\in A^2$. Then by Theorem 8.2, A^* is a subDGA of $dG^*(G/\Gamma,\omega)$ and the inclusion $A^*\subset dG^*(G/\Gamma,\omega)$ induces a cohomology isomorphism. As the proof of Theorem 8.2, considering

$$D^*_\Gamma=\left\langle\frac{\partial}{\partial z_I}\wedge\frac{\partial}{\partial w_J}\wedge\frac{\partial}{\partial\bar{z}_K}\wedge\frac{\partial}{\partial\bar{w}_L}\middle|(\alpha_J\bar{\alpha}_L)_{|\Gamma}=1\right\rangle,$$

then we have an isomorphism $D^*_\Gamma\cong A^*_\Gamma$ of DGAs and hence the bracket $[\bullet]_\omega$ on the DGA A^*_Γ is 0.

We have:

Theorem 9.2. *Let $G=\mathbb{C}^n\ltimes_\phi\mathbb{C}^m$ with a semi-simple unimodular action $\phi:\mathbb{C}^n\to GL_m(\mathbb{C})$ (not necessarily holomorphic). We suppose G has a lattice Γ. We assume the condition $(\mathcal{C}_r)$ in Remark 4 holds for any $r\in\mathbb{N}$. Then we have*

- *$\mathcal{H}^*(G/\Gamma,J)$ is a sub-DGA of $dG^*(G/\Gamma,J)$ with trivial brackets. In particular, $Kur^{gen}(G/\Gamma,J)$ is smooth.*
- *Let $\mu\in C^{2,0}_\Gamma$ be a holomorphic poisson bi-vector field. Then the holomorphic poisson cohomology $H^*(G/\Gamma,\mu)$ is isomorphic to the Dolbeault cohomology $H^*(dG^*(G/\Gamma,J))$.*
- *We assume G/Γ admits a symplectic structure ω and $\omega\in A^2_\Gamma$. Then the DGAs $dG^*(G/\Gamma,J)$ and $dG^*(G/\Gamma,\omega)\otimes\mathbb{C}$ are quasi-isomorphic.*

Proof. We have

$$C^{p,q}_\Gamma=\left\langle\frac{\partial}{\partial z_I}\wedge\frac{\partial}{\partial w_J}\wedge d\bar{z}_K\wedge d\bar{w}_L\middle|\begin{array}{c}|I|+|K|=p,\ |J|+|L|=q\\(\alpha_J\bar{\alpha}_L)_{|\Gamma}=1\end{array}\right\rangle.$$

Hence the first assertion follows.

For a a holomorphic poisson bi-vector field $\mu\in C^{2,0}_\Gamma$, since $C^{p,q}$ is written as above, we can write

$$\mu=\sum_{|I|+|J|=2}a_{IJ}\frac{\partial}{\partial z_I}\wedge\frac{\partial}{\partial w_J}$$

for $a_{IJ}\in\mathbb{C}$. Then the operator $[\mu\bullet]:C^{*,*}_\Gamma\to C^{*,*+1}_\Gamma$ is trivial. Hence the second assertion follows.

We assume G/Γ admits a symplectic structure ω and $\omega\in A^2_\Gamma$. Then we have $A^*_\Gamma=C^*_\Gamma$ and the two DGAs have the trivial bracket. Hence the second assertion follows. $\square$

Example 4. Let $G=\mathbb{C}\ltimes_{\phi}\mathbb{C}^{2m}$ such that

$$\phi(x+\sqrt{-1}y))=\mathrm{diag}(e^{a_1x},e^{-a_1x},\dots,e^{a_mx},e^{-a_mx})$$

for integers $a_i\neq 0$. Then we have a left-invariant pseudo-Kähler structure

$$\omega=\sqrt{-1}dz\wedge d\bar{z}+\sum_{i=1}^{m}(dw_{2i-1}\wedge d\bar{w}_{2i}+d\bar{w}_{2i-1}\wedge dw_{2i}).$$

We have $\beta_{2i-1}=e^{-2a_i\pi\sqrt{-1}y}$, $\beta_{2i}=e^{2a_i\pi\sqrt{-1}y}$. We can write $G=\mathbb{R}\times(\mathbb{R}\ltimes_{\phi}\mathbb{C}^{2m})$. G has a lattice $\Gamma=t\mathbb{Z}\times\Delta$ where Δ is a lattice $\mathbb{R}\ltimes_{\phi}\mathbb{C}^{2m}$ for $t>0$. Then $B^{p,q}$ varies for the choice $t>0$. Consider the case $t\neq\frac{r}{s}\pi$ for any $r,s\in\mathbb{Z}$. Then the condition $(\mathcal{C}_r)$ in Remark 4 holds for any $r\in\mathbb{N}$. Hence any small generalized deformation is a left-invariant generalized complex structure on G/Γ, $Kur^{gen}(G/\Gamma,J)$ is smooth and $dG^{*}(G/\Gamma,J)$ and $dG^{*}(G/\Gamma,\omega)\otimes\mathbb{C}$ are quasi-isomorphic.

Consider the new Lie group $G'=\mathbb{C}\times G$ and its lattice $\Gamma'=(a\mathbb{Z}+b\sqrt{-1}\mathbb{Z})\times\Gamma$. We have a holomorphic poisson bi-vector field

$$\mu=\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial z'}+\sum_{i=1}^{m}\frac{\partial}{\partial w_{2i-1}}\wedge\frac{\partial}{\partial w_{2i}}$$

which is given by a holomorphic symplectic form for a coordinate z' of the additional $\mathbb{C}$. Then we have

$$H^{*}(G'/\Gamma',\mu)\cong H^{*}(dG^{*}(G'/\Gamma',J).$$

Example 5. Consider $G=\mathbb{C}^{n}\ltimes_{\phi}\mathbb{C}^{2n+2}$ such that

$$\phi(z_1,\dots,z_n)=\mathrm{diag}(e^{x_1},\dots,e^{x_n},e^{-x_1-\dots-x_n},e^{-x_1},\dots,e^{-x_n},e^{x_1+\dots+x_n})$$

for complex coordinate

$$(z_1=x_1+\sqrt{-1}y_1,\dots,z_n=x_n+\sqrt{-1}y_n,w_1,\dots,w_{n+1},w_{n+2},\dots,w_{2n+2})$$

. Then we have the pseudo-Kähler structure

$$\omega=\sum dz_i\wedge d\bar{z}_i+\sum dw_i\wedge d\bar{w}_{n+1+i}+d\bar{w}_i\wedge dw_{n+1+i}.$$

By a totally real algebraic number field K with degree $n+1$, we have a lattice $\Gamma=(\Gamma'_1+\Gamma'_2)\ltimes\Gamma''$ where Γ'_1 is a lattice of $\mathbb{R}^n\subset\mathbb{C}^n$ which is constructed by the group of the units of algebraic integers in K (see [25]) and Γ'_2 is a lattice of $\sqrt{-1}\mathbb{R}^n\subset\mathbb{C}^n$.

$$\begin{aligned}&(\beta_1,\dots,\beta_n,\beta_{n+1},\beta_{n+2},\dots,\beta_{2n+1},\beta_{2n+2})\\&\quad=(e^{\sqrt{-1}y_1},\dots,e^{\sqrt{-1}y_n},e^{-\sqrt{-1}y_1-\dots-\sqrt{-1}y_n},e^{-\sqrt{-1}y_1},\dots,e^{-\sqrt{-1}y_n},e^{\sqrt{-1}y_1+\dots+\sqrt{-1}y_n}).\end{aligned}$$

By this we can take Γ'_2 such that We assume the condition $(\mathcal{C}_r)$ in Remark 4 holds for any $r\in\mathbb{N}$. For example, $\Gamma'_2=\sqrt{-1}\mathbb{Z}^n$. Hence for such Γ, $Kur^{gen}(G/\Gamma,J)$ is smooth and $dG^{*}(G/\Gamma,J)$ and $dG^{*}(G/\Gamma,\omega)\otimes\mathbb{C}$ are quasi-isomorphic.

Remark 6. In [9], Hasegawa showed that a simply connected solvable Lie group G with a lattice Γ such that G/Γ admits a Kähler structure can be written as $G=\mathbb{R}^{2k}\ltimes_{\phi}\mathbb{C}^{l}$ such that

$$\phi(t_j)((z_1,\dots,z_l))=(e^{\sqrt{-1}\theta_1^{j}t_j}z_1,\dots,e^{\sqrt{-1}\theta_l^{j}t_j}z_l),$$

where each $e^{\sqrt{-1}\theta_i^{j}}$ is a root of unity. In particular if G is completely solvable and a solvmanifold G/Γ admits a Kähler structure, then G is abelian. Hence Example 4 and Example 5 do not admit Kähler structures.

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